Lecture 32

Correction $R_{x_0}(u,v) = -\mathrm{ad}_{[u,v]}$
Sectional $R(u,v)(v,u)$

Last time: $R_{x_0}(u,v)(w,x) = -B([[u,v],w],x)$ $\quad u,v,w,x \in \mathfrak{p}$.

killing form

<u>Thm.</u> If X is a symmetric space of noncpt type, then X is nonpositively curved.

<u>Pf.</u> As X has a transitive isometric group action, it suffices to show $R_{x_0}(u,v)(u,v) \leq 0 \quad \forall\, u,v \in \mathfrak{p}$.

$$R_{x_0}(u,v)(u,v) = B([[u,v],u],v) = -B([u,[u,v]],v)$$

Now B is Ad_G-inv't, i.e. $B(\mathrm{Ad}_g x, \mathrm{Ad}_g y) = B(x,y)$
$\Rightarrow$ ad-inv't, i.e. $B([t,x],y) + B(x,[t,y]) = 0$

$$-B([u,[u,v]],v) \overset{*}{=} B([v,v],[u,v]) \leq 0$$ because $[u,v] \in \mathfrak{k}$, $B|_{\mathfrak{k}}$ neg def. $\square$

BTW: compact $\Rightarrow$ neg def is really

compact $\Rightarrow \exists$ bi-inv't Riem metric on G $\Rightarrow \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_x) = \sum_i \langle \mathrm{ad}_x \mathrm{ad}_x v_i, v_i \rangle$
$$\underset{\text{bi invar}}{=} -\sum_i \langle \mathrm{ad}_x v_i, \mathrm{ad}_x v_i \rangle \leq 0$$

(It can also be shown that X of cpt type is non neg curved.)

<u>Geodesics</u> $\forall\, x \in X,\ v \in T_x X\ \exists!\ \gamma: J \to X$ geod w/ $\gamma'(0) = v$.
For X complete, take $J = \mathbb{R}$.

<u>Exponential map</u> Riemannian exp map is
$$\mathrm{Exp}_x: T_x X \longrightarrow X: \quad \mathrm{Exp}_x(v) = \gamma(1) \text{ where } \gamma \text{ geod}, \gamma'(0) = v.$$
(we've phrased it for X complete)

If γ is a geod, so is $t \mapsto \gamma(kt)$ $\forall k \in \mathbb{R}$.
$\Rightarrow$ $t \mapsto \mathrm{Exp}_x(tv)$ is the geod through (x,v).

Prop Exp_x is smooth, $(d\mathrm{Exp}_x)_x = \mathrm{Id}_{T_xX}$.

Theorem (Cartan-Hadamard) If X is a complete, nonpos curved, simply conn. Riem manifold, then Exp_x is a diffeomorphism $\forall x \in X$.

We already know G/K is contractible as $G = KS$ where K max cpt and S solvable ($\Rightarrow$ contractible).

Now we further know Exp_x is one such.

Cor. G/K is uniquely geodesic, ie. any two points are connected by a unique geodesic which is also minimizing.
(Uses Riem fact: complete $\Rightarrow$ every pair joined by min geodesic seg)

Fact. $s_p \circ s_q$ preserves $\overleftrightarrow{pq}$ and moves along it by $d(p,q)$

OK, so what is Exp for a symmetric space?

let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomp assoc w/ a symm space G/K.
$x_0 \in G/K$ base point.

Thm (Helgason thm 3.3). If $\pi: G \to G/K$ is the projection, $\forall u \in \mathfrak{p}$ we have

$$\pi(\exp(tu)) = \mathrm{Exp}_{x_0}(tu).$$

$$\exp(tu)\cdot K = \exp(tu)\cdot x_0$$

That is, every geodesic of G/K through the basepoint is simply the orbit of the basept under 1-param group of isometries assoc to $u \in \mathfrak{p}$.

Note: In general, $\exp(tX)\cdot p$ is not a geodesic!

$\{\exp(u) \mid u \in \mathfrak{p}\}$ is called the set of transvections at x_0.

$f \circ \_ \circ f^{-1}$ ______________ at $f(x_0)$

So $\underbrace{\mathrm{Stab}_p}_{\text{grp}}$ and $\underbrace{\mathrm{Transvec}_p}_{\text{not grp}}$ are "complementary" sets of isom.

Note: Given $p,q \in X$ $\exists!$ transvection $p \to q$

Totally geod. X Riemannian $S \subset X$ inherits metric $g|_{TS}$

A submfld $S \subset X$ is called totally geodesic if every geodesic of $(S, g|_{TS})$ is in fact a geodesic of X.

e.g. a geodesic is a 1d tot geod sub.

Lie triple sys. of Lie alg. $\mathfrak{m} \subset \mathfrak{g}$ vec subsp.

$\mathfrak{m}$ is a triple system if $[[\mathfrak{m},\mathfrak{m}],\mathfrak{m}] \subset \mathfrak{m}$.

Note: $[\mathfrak{m},\mathfrak{m}]$ need not lie in $\mathfrak{m}$!

Ex. $\mathfrak{p} \subset \mathfrak{g}$ is a Lie triple system

Thm. $X = G/K$ sym space, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then if $\mathfrak{m} \subset \mathfrak{g}$ is a Lie triple system and $\mathfrak{m}$ is a subspace of $\mathfrak{p}$, then

$$S = \exp(\mathfrak{m}) \cdot x_0$$

is a complete totally geod submfld of X. Every such submfld through x_0 has this form.

Sketch. Turns out to be enough to show

$R_{x_0}(u,v)w \in T_{x_0}S$ if $u,v,w \in T_{x_0}S$ to show S is TG.

But that's just $[[u,v],w] \in \mathfrak{m}$. Subtle: This is only enough because $\nabla R = 0$ for symmetric spaces. $\square$

Note. S is also symmetric.

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra.
(e.g. if G cplx, K the cpt real form assoc to G, H.
Let $\mathfrak{a} \subset \mathfrak{h}$ be the subspace where roots are real.)

Thm. $\exp(\mathfrak{a}) \cdot x_0$ is isometric to $\mathbb{E}^r$ where $r = \dim \mathfrak{a}$.
These are exactly the maximal isometrically embedded Eucl spaces
and every Eucl tot geod
is contained in one.

Let's work out an example of this for $SL_n\mathbb{R}$.